Lars Birkedal
Daniel Gratzer

Assignment 5
Hand in date: Wed Nov 2

Definition 1. Let $F : \mathbb{C} \to \mathbb{C}$ be a functor.

- A *fixed point* of the functor F is an object X such that $F(X) \cong X$.
- An *algebra* for the functor F is a pair (X, ϕ) where X is an object of $\mathbb{C}$ and $\phi : FX \to X$ is a morphism. The object X is called the *carrier* of the algebra.

  The algebra (L, γ) is *initial* if for any other algebra (X, ϕ) there exists a *unique morphism* f such that the following diagram commutes

$$\begin{tikzcd} F(L) \arrow[r, "F(f)"] \arrow[d, "\gamma"] & F(X) \arrow[d, "\phi"] \\ L \arrow[r, "f"] & X \end{tikzcd}$$

- A *coalgebra* for the functor F is a pair (X, ϕ) where X is an object of $\mathbb{C}$ and $\phi : X \to FX$ is a morphism. The object X is called the *carrier* of the coalgebra.

  The coalgebra (L, γ) is *final* if for any other coalgebra (X, ϕ) there exists a *unique morphism* f such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"] & L \arrow[d, "\gamma"] \\ F(X) \arrow[r, "F(f)"] & F(L) \end{tikzcd}$$

Remark 1. The dual concepts, of a final *algebra* and initial coalgebra, are not particularly useful.

Exercise 1. Show that if (L, γ) and (L', γ') are initial algebras for F then there exists a unique *isomorphism* f such that

$$\begin{tikzcd} F(L) \arrow[r, "F(f)"] \arrow[d, "\gamma"] & F(L') \arrow[d, "\gamma'"] \\ L \arrow[r, "f"] & L' \end{tikzcd}$$

commutes, i.e., initial algebras are unique up to isomorphism.

Formulate and prove an analogous result for final coalgebras.

Exercise 2. Show that if (L,γ) is an initial algebra (resp. final coalgebra) for the functor F then γ is an *isomorphism*.
Hint: $(F(L),F(\gamma))$ is also an algebra (resp. coalgebra) for F.

Remark 2. The previous exercise shows that initial algebras and final coalgebras of F are in particular its fixed points.

Exercise 3. Some functors have neither an initial algebra nor a final coalgebra.

- Show that the power set functor described in the first assignment has no fixed point. Recall that this functor maps the set X to its power set $\mathcal{P}(X)$ and it maps a function $f : X \to Y$ to the image function $\mathcal{P}(X) \to \mathcal{P}(Y)$.
- Conclude that it has neither an initial algebra nor a final coalgebra.

Definition 2. A *colimit of type* ω is a colimit of a diagram of type $(\mathbb{N},\leq)$ where $(\mathbb{N},\leq)$ is the poset of natural numbers with less than or equal relation considered as a category.

A *limit of type* ω^{op} is a limit of a diagram of type $(\mathbb{N},\geq)$ where $(\mathbb{N},\geq)$ is the poset of natural numbers with greater than or equal relation considered as a category.

Exercise 4. Suppose $\mathbb{C}$ has an initial object 0 and colimits of type ω. Suppose $F : \mathbb{C} \to \mathbb{C}$ preserves colimits of type ω. Define the sequence of objects F_n and arrows $i_n : F_n \to F_{n+1}$ as follows.

$$F_0 = 0 \qquad\qquad i_0 = !_{F(0)}$$
$$F_{n+1} = F(F_n) \qquad\qquad i_{n+1} = F(i_n)$$

These thus define the following diagram

$$F_0 \xrightarrow{i_0} F_1 \xrightarrow{i_1} F_2 \xrightarrow{i_2} F_3 \xrightarrow{i_3} F_4 \xrightarrow{i_4} \cdots$$

Show that the *colimit* L of this diagram is the carrier of the initial algebra for F. This means that you need to define a map $\gamma : F(L) \to L$ and show it satisfies the universal property described in Definition 1.

Exercise 5. Suppose $\mathbb{C}$ has a terminal object 1 and limits of type ω^{op}. Suppose $F : \mathbb{C} \to \mathbb{C}$ preserves limits of type ω^{op}. Define the sequence of objects F_n and arrows $p_n : F_{n+1} \to F_n$ as follows.

$$F_0 = 1 \qquad\qquad p_0 = !_{F(0)}$$
$$F_{n+1} = F(F_n) \qquad\qquad p_{n+1} = F(p_n)$$

These thus define the following diagram

$$F_0 \xleftarrow{p_0} F_1 \xleftarrow{p_1} F_2 \xleftarrow{p_2} F_3 \xleftarrow{p_3} F_4 \xleftarrow{p_4} \cdots$$

Show that the *limit* L of this diagram is the carrier of the final coalgebra algebra for F.

Hint: Use duality.

Remark 3. In lecture we shall show that these initial algebras can be used to define inductive types, and the universal property defines the structural recursion principle familiar to functional programmers.
