Category Theory
Fall 2022

Lars Birkedal
Daniel Gratzer

Assignment 4
Hand in date: Wed Oct 26

Please skip exercises 1 and 2; they deal with "equivalence" which we haven't yet covered.

Exercise 1. Show that any preorder is equivalent (as a category) to a poset.

Exercise 2. Show that a cartesian closed category with a *zero object* (recall this is an object which is both initial and terminal) is *equivalent* to the category with exactly one object and one arrow, the identity.

Exercise 3. Let $\mathbb{C}$ be a cartesian closed category. Show the following properties.

- Let $f : A \times B \to C$ and $g : C \to D$ be morphisms. Recall that using exponential transposes and evaluation we can define the morphism $g^B : C^B \to D^B$. Show

$$g^B \circ \widetilde{f} = \widetilde{g \circ f}$$

  as morphisms $A \to D^B$.

- Show that for any $f : A \times B \to C$ and any $h : A' \to A$ we have

$$\widetilde{f} \circ h = f \circ \widetilde{(h \times \mathrm{id})}$$

  as morphism $A' \to C^B$.

Exercise 4. Let $\mathbb{C}$ be a cartesian closed category. Recall that in such a category the mapping $f \mapsto \widetilde{f}$ is an isomorphism (bijection) of hom-sets

$$\mathbf{Hom}_{\mathbb{C}}(A \times B, C) \to \mathbf{Hom}_{\mathbb{C}}\left(A, C^B\right).$$

Let us call this isomorphism $\Lambda_{A,B,C}$. Show that it is natural in C and A.

Concretely this means that you must show that the following diagram commutes for any morphism $g : C \to D$

$$\begin{array}{ccc}
\mathbf{Hom}_{\mathbb{C}}(A \times B, C) & \xrightarrow{\Lambda_{A,B,C}} & \mathbf{Hom}_{\mathbb{C}}\left(A, C^B\right) \\
{\scriptstyle \mathbf{Hom}_{\mathbb{C}}(A \times B, g)}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{Hom}_{\mathbb{C}}(A, g^B)} \\
\mathbf{Hom}_{\mathbb{C}}(A \times B, D) & \xrightarrow{\Lambda_{A,B,D}} & \mathbf{Hom}_{\mathbb{C}}\left(A, D^B\right)
\end{array}$$

and that the following diagram commutes for any morphism $h : A' \to A$.

$$\begin{array}{ccc}
\mathbf{Hom}_{\mathbb{C}}(A \times B, C) & \xrightarrow{\Lambda_{A,B,C}} & \mathbf{Hom}_{\mathbb{C}}\left(A, C^B\right) \\
{\scriptstyle \mathbf{Hom}_{\mathbb{C}}(h \times \mathrm{id}, C)} \downarrow & & \downarrow {\scriptstyle \mathbf{Hom}_{\mathbb{C}}(h, C^B)} \\
\mathbf{Hom}_{\mathbb{C}}(A' \times B, C) & \xrightarrow{\Lambda_{A',B,C}} & \mathbf{Hom}_{\mathbb{C}}\left(A', C^B\right)
\end{array}$$

Remark 1. In brief, this shows that the two functors

$$(A, C) \mapsto \mathbf{Hom}_{\mathbb{C}}(A \times B, C)$$

and

$$(A, C) \mapsto \mathbf{Hom}_{\mathbb{C}}\left(A, C^B\right)$$

are isomorphic as functors $\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Sets}$. Later on we shall see that this means precisely that the functor $A \mapsto A \times B$ is *left adjoint* to the functor $C \mapsto C^B$.
