Assignment 3
Hand in date: Wed Oct 5

Exercise 1. Suppose the square

$$
\begin{array}{ccc}
A & \xrightarrow{\ h\ } & B \\
{\scriptstyle k}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\
C & \xrightarrow[\ f\ ]{} & D
\end{array}
$$

is a pullback. Show the following.

1. If g is a *split epimorphism* then k is.

2. If g is an *isomorphism* then k is.

3. If g is a *split monomorphism* then k is not necessarily a split monomorphism.

Exercise 2. Recall that the two-pullback lemma, or the pullback pasting lemma, states that in a commutative diagram

$$
\begin{array}{ccccc}
A & \longrightarrow & B & \longrightarrow & C \\
\downarrow & & \downarrow & & \downarrow \\
D & \longrightarrow & E & \longrightarrow & F
\end{array}
$$

- if the right square and outer rectangle are pullbacks then the left square is, and

- if the left and right squares are pullbacks so is the outer rectangle.

Show that if the left square and the outer rectangle are pullbacks then the right square can fail to be a pullback.

Exercise 3. A *regular monomorphism* is an arrow $e : E \to A$ which is an equalizer of some pair of arrows $f, g : A \to B$. Recall that by Proposition 3.16 of SA e is in particular a monomorphism.

Show that in a pullback square

$$
\begin{array}{ccc}
E' & \longrightarrow & E \\
{\scriptstyle e'}\downarrow & \lrcorner & \downarrow{\scriptstyle e} \\
C & \xrightarrow[\ h\ ]{} & A
\end{array}
$$

if e is a regular monomorphism then so is e' for any object C and any arrow h.

This property is often called "regular monos are stable under pullback".

Exercise 4. Let $e : A \to B$ be a *regular monomorphism*. Show that if the square

$$
\begin{array}{ccc}
A & \xrightarrow{\ e\ } & B \\
{\scriptstyle e}\downarrow & & \downarrow{\scriptstyle y} \\
B & \xrightarrow{\ x\ } & Q
\end{array}
$$

is a *pushout* then e is the equalizer of x and y.

Exercise 5. Let $F : \mathbb{C} \to \mathbb{D}$ be a finite limit preserving functor. Show that for any monomorphism $m : A \to B$ the morphism $F(m) : F(A) \to F(B)$ is also a monomorphism, i.e., F preserves monomorphisms.

Dualizing, show that if $F : \mathbb{C} \to \mathbb{D}$ preserves finite colimits then it preserves epimorphisms.

Exercise 6. Give an example each of a functor **Sets** $\to$ **Sets** that:

1. Both preserves and creates terminal objects;

2. Preserves, but does not create, terminal objects;

3. Neither preserves nor creates terminal objects.

Finally show that any functor **Sets** $\to$ **Sets** which creates terminal objects also preserves them.

Exercise 7 (Not mandatory). Try to come up with an example of a functor which does not preserve monomorphisms, and an example of a functor which does not preserve epimorphisms.