Category Theory
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Assignment 1
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Exercise 1. Let T_0 send a set X to its power set $\mathcal{P}(X)$, and let T_1 send a function $f : X \to Y$ to the image function

$$T_1(f) : T_0X \to T_0Y,$$

which is defined as

$$T_1(f)(A) = \{f(x) \mid x \in A\}$$

Show that $T = (T_0, T_1)$ is a functor from **Sets** to **Sets**.

Exercise 2. Define the category $\mathbb{K}$ as follows. Its objects are sets. Morphisms $X \to Y$ in $\mathbb{K}$ are morphisms $X \to T(Y)$ in **Sets**, i.e.,

$$\mathbf{Hom}_{\mathbb{K}}(X, Y) = \mathbf{Hom}_{\mathbf{Sets}}(X, T(Y)).$$

Composition is defined as follows: if $f : X \to Y$ and $g : Y \to Z$ are two morphisms in $\mathbb{K}$ then

$$(g \circ f)(x) = \bigcup_{y \in f(x)} g(y) = \{z \mid \exists y \in f(x), z \in g(y)\}.$$

- Show that $\mathbb{K}$ is a category.
- Show that it is isomorphic to the category of sets and relations **Rel**.
  Hint: Any subset $R \subseteq X \times Y$ can be represented as a function $F(R) : X \to \mathcal{P}(Y)$ defined as
  $$F(R)(x) = \{y \mid (x, y) \in R\}.$$

Exercise 3. Let $\mathbb{C}$ be a category with binary products.

- Is the projection $\pi_X : X \times Y \to X$ an epimorphism in general? Is it a monomorphism?
- Let $f : Z \to X$, $g : Z \to Y$, and $h : W \to Z$ be three morphisms. Show
  $$\langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle$$
  as morphisms $W \to X \times Y$.
- Let $f : Z \to X$ and $g : W \to Y$ be two morphisms. Show there *exists* a *unique* morphism $u : Z \times W \to X \times Y$ such that for all objects A and morphisms $h_Z : A \to Z$ and $h_W : A \to W$
  $$u \circ \langle h_Z, h_W \rangle = \langle f \circ h_Z, g \circ h_W \rangle.$$
  as morphisms $A \to X \times Y$. This unique morphism u is typically written as $f \times g$.
- Using the notation from the previous item, show that for any morphisms $f : Z \to X, g : W \to Y$, $h : A \to Z$, and $k : B \to W$ we have
  $$(f \times g) \circ (h \times k) = (f \circ h) \times (g \circ k)$$
  as morphisms $A \times B \to X \times Y$.